The Virasoro algebra

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this is an incomplete draft

"Given a classical theory with this conformal group as symmetry group, one studies the group $\text{Diff}_+(\mathbb{S})$ and its Lie algebra first. After quantization one is interested in the unitary representations of the central extensions of $\text{Diff}_+(\mathbb{S})$ or Lie $(\text{Diff}_+(\mathbb{S}))$ in order to get representations in the Hilbert space."

$\text{Diff}_+(\mathbb{S})$ is the group of orientation preserving diffeomorphism of the circle.

We set $\text{Lie}(\text{Diff}_+(\mathbb{S})) := \text{Vect}(\mathbb{S})$, the real vector space of smooth vector fields on $\mathbb{S}$.

An element of this space is an R-linear derivation of $C^\infty(\mathbb{S})$ functions and the Lie bracket is the commutator. The space of $C^\infty(\mathbb{S})$ functions is simply the C^∞ functions on $\mathbb{R}$ which have a period of 2π, denoted $C^\infty_{2\pi}(\mathbb{R})$.

Representing points on $\mathbb{S}$ as $z = e^{i\theta}$ (θ being a real variable) a vector field on $\mathbb{S}$ is of the $f\frac{d}{d\theta}$, where f is a $C^\infty_{2\pi}(\mathbb{R})$ function. The fourier series of f,

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta) \tag{0.1}$$

provides topological generators for this vector space, $\frac{d}{d\theta}$, $cos(n\theta)\frac{d}{d\theta}$ and $sin(n\theta)\frac{d}{d\theta}$. The complexification of $Vect(\mathbb{S})$ seems to be of interest. (I don't yet know why.) The restricted Lie algebra of polynomial vector fields on $\mathbb{S}$ is called the Witt algebra W. Define

$$L_n := z^{1-n}\frac{d}{dz} \tag{0.2}$$

for $n \in \mathbb{Z}$. We define the Witt algebra

$$W := \mathbb{C}L_n \tag{0.3}$$

Defining the Lie bracket as the commutator, a quick calculation yields

$$[L_n, L_m] = (n-m)L_{n+m} \tag{0.4}$$

Theorem 0.1. *$H^2(W, \mathbb{C}) = \mathbb{C}$.*

This theorem was proved by Gelfand-Fuchs in 1968.

Proof. Define $\omega : W \times W \to \mathbb{C}$ defined by

$$\omega(L_n, L_m) = \frac{n}{12}(n^2 - 1)\delta_{n+m} \tag{0.5}$$

where δ_k equals 1 when $k = 0$ and is 0 otherwise. We will show that this is (upto equivalence) the only possible central extension of the Witt algebra. ω is clearly bilinear and alternating. First, we need to check that $\omega \in Z^2(W, \mathbb{C})$, for which the condition is,

$$\omega(L_k, [L_m, L_n]) + \omega(L_m, [L_n, L_k]) + \omega(L_n, [L_k, L_m]) = 0. \tag{0.6}$$

This condition may be verified using the definition of ω. The details are left to the reader.

Secondly, we need to prove that ω is not a coboundary, i.e., $\omega \notin B^2(W, \mathbb{C})$. We'll prove this by contradiction. Let $\mu \in \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$ such that $\omega(X, Y) = \mu([X, Y])$ for all $X, Y \in W$.

Then for every $n \in \mathbb{Z}$ we must have

$$\omega(L_n, L_{-n}) = \mu([L_n, L_{-n}]) \tag{0.7}$$

$$\frac{n}{12}(n^2 - 1) = 2n\mu(L_0) \tag{0.8}$$

$$\mu(L_0) = \frac{(n^2 - 1)}{24} \tag{0.9}$$

This is clearly not true for all $n \in \mathbb{Z}$. Hence $\omega \notin B^2(W, \mathbb{C})$.

Lastly, if Θ is another central extension of W, then we need to show it is linearly dependent on ω.

$$\Theta(L_k, [L_m, L_n]) + \Theta(L_m, [L_n, L_k]) + \Theta(L_n, [L_k, L_m]) = 0$$
$$(m - n)\Theta(L_k, L_{m+n}) + (n - k)\Theta(L_m, L_{n+k}) + (k - m)\Theta(L_n, L_{k+m}) = 0$$

Setting $k = 0$, we get

$$(m - n)\Theta(L_0, L_{m+n}) + n\Theta(L_m, L_n) - m\Theta(L_n, L_m) = 0$$

which yields

$$\Theta(L_0, L_{m+n}) = \frac{(m + n)}{(m - n)}\Theta(L_m, L_n)$$

for $m \neq -n$. Define a homomorphism $\mu : W \to \mathbb{C}$,

$$\mu(L_n) = \frac{1}{n}\Theta(L_0, L_n)$$

for $n \in \mathbb{Z}\backslash\{0\}$.

$$\mu(L_0) = -\frac{1}{2}\Theta(L_1, L_{-1})$$

Denote by $\tilde{\mu}$ a coboundary in $Z^2(W, \mathbb{C})$ and let $\Theta' := \Theta + \tilde{\mu}$. Then for $m, n \in \mathbb{Z}$ and $m \neq -n$,

$$\Theta'(L_n, L_m) = \Theta(L_n, L_m) + \mu([L_n, L_m]) \tag{0.10}$$

$$\Theta'(L_n, L_m) = \frac{m-n}{m+n}\Theta(L_0, L_{n+m}) + \mu((n-m)L_{n+m}) \tag{0.11}$$

$$\Theta'(L_n, L_m) = \frac{m-n}{m+n}\Theta(L_0, L_{n+m}) + \frac{n-m}{m+n}\Theta(L_0, L_{n+m}) \tag{0.12}$$

$$\Theta'(L_n, L_m) = 0 \tag{0.13}$$

This means there's a map $h : \mathbb{Z} \to \mathbb{C}$ such that for $n, m \in \mathbb{Z}$

$$\Theta'(L_n, L_m) = \delta_{m+n} h(n) \tag{0.14}$$

Since Θ' is alternating $h(-k) = -h(k)$.

$$\begin{aligned} h(1) &= \Theta'(L_1, L_{-1}) \\ &= \Theta(L_1, L_{-1}) + \mu([L_1, L_{-1}]) \\ &= \Theta(L_1, L_{-1}) + \mu(2L_0) \\ &= \Theta(L_1, L_{-1}) - \Theta(L_1, L_{-1}) \\ &= 0. \end{aligned}$$

Using the properties of Θ' we'd like to derive a recursive formula for $h(n)$.

$$\begin{aligned} \Theta(L_k, [L_m, L_n]) + \Theta'(L_m, [L_n, L_k]) + \Theta'(L_n, [L_k, L_m]) &= 0 \\ (m-n)\Theta(L_k, L_{m+n}) + (n-k)\Theta(L_m, L_{n+k}) + (k-m)\Theta(L_n, L_{k+m}) &= 0 \end{aligned}$$

For $k + m + n = 0$,

$$\begin{aligned} (m-n)h(k) + (n-k)h(m) + (k-m)h(n) &= 0 \\ -(m-n)h(m+n) + (2n+m)h(m) - (2m+n)h(n) &= 0 \end{aligned}$$

Letting $n = 1$,

$$-(m-1)h(m+1) + (2+m)h(m) - (2m+1)h(1) = 0$$

$$h(m+1) = \frac{2+m}{m-1}h(m) \tag{0.15}$$

where $m \in Z\backslash\{1\}$.

We now boldly claim that $h(m) = \frac{\lambda}{12}m(m^2-1)$, where $\lambda \in \mathbb{C}^*$. By induction

$$\begin{aligned}
h(m+1) &= \frac{\lambda(2+m)}{12(m-1)}m(m^2-1) \\
h(m+1) &= \frac{\lambda(2+m)}{12}m(m+1) \\
h(m+1) &= \frac{\lambda}{12}(m+1)m(m+2) \\
h(m+1) &= \frac{\lambda}{12}(m+1)((m+1)^2-1)
\end{aligned}$$

Hence $\Theta' = \lambda\omega$. □

The Virasoro algebra is the unique central extension of the Witt algebra.

$$\mathrm{Vir} = W \oplus \mathbb{C}Z \tag{0.16}$$

where $[L_n, Z] = 0$ for all $n \in \mathbb{Z}$.