THE VIRASORO ALGEBRA

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Introduction

An extension of a group H by a group G is a short exact sequence

$$1 \to G \to \hat{G} \to H \to 1$$

which is exact at each position, i.e., the image of the preceding map is the kernel of the next map. The extension is central if the image of G in $\hat{G}$ is central.

Examples of extensions:

- The trivial extension.

$$1 \to G \to G \times H \to H \to 1$$

- The semidirect product.

$$1 \to G \to G \ltimes H \to H \to 1$$

  This differs from the trivial extension by that there is a homomorphism $G \to \mathrm{Aut}(H)$.
- Affinization of a vector space.

$$1 \to \mathrm{GL}(V) \to \mathrm{GL}(V) \ltimes V \to V \to 0$$

  This is a special case of the above example.
- Universal covering group of SO(1,3).
  The universal cover of SO(1,3) is isomorphic to a central extnsion of SO(1,3) by $\mathbb{Z}/2\mathbb{Z}$.

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(1{,}3) \to 0$$

  In general, the universal cover E over a Lie group G is isomorphic to an extension of G by $\pi_1(G)$, its fundamental group. The fundamental group is then the group of deck transformations of G, i.e., the subgroup of $\mathrm{Aut}(E)$ that leaves G fixed.
- Projective general linear group

$$1 \to k^{\times} \to \mathrm{GL}(V) \to \mathrm{PGL}(V) \to 0$$

  This is also a central extension.

Equivalence of extensions

Consider two extensions

$$E : 1 \to A \to E \to G \to 1$$

$$E' : 1 \to A \to E' \to G \to 1$$

Two extensions E, E' are equivalent if there is an isomorphism $\psi : E \to E'$ such that the map (id_A, ψ, id_G) has all squares commutative.

An exact sequence E (as above) splits if there is a homomorphism $\sigma : G \to E$ such that $\pi \circ \sigma = id_G$.

Theorem 0.1. *An extension splits if and only if it is equivalent to the trivial extension.*

Proof. If the extension E splits, define the isomorphism

$$\psi : A \times G \to E$$

$$(a, g) \mapsto \iota(a)\pi(g)$$

where ι and π are the injection and surjection maps of the extension.
If there is an isomorphism $\psi : A \times G \to E$, then define

$$\sigma(g) = \psi(1, g)$$

where $1 \in A$ is the unit element. □

Consider a central extension E as above, and define a map

$$\tau : G \to E$$

such that $\tau \circ \pi = id_G$ and $\tau(1) = 1$. Let $\tau_x : \tau(x)$ for $x \in G$.
Define

$$\omega : G \times G \to A$$

$$(g, h) \mapsto \tau_g \tau_h \tau_{gh}^{-1}.$$

This map lets us measure to what extent the extension "fails" to be a trivial extension.

ω is actually a *two-cocycle*. It satisfies the following properties which can be easily checked:

(1) $\omega(1, 1) = 1$
(2) $\omega(x, yz)\omega(y, z) = \omega(xy, z)\omega(x, y)$

for $x, y, z \in G$.

Any central extension may be written in the form

$$1 \to A \to A \times_\omega G \to G \to 1.$$

The multiplication in $A \times_\omega G$ is given by $(a, x)(b, y) := (\omega(x, y)ab, xy)$.

This yields a correspondence between the set of extensions of G by A, and the set of cocycles on G with values in A.

The second cohomology group $H^2(G, A)$ is the set of all 2-cocycles on G with values in A modulo an equivalence relation $\sim$. $\omega \sim \omega'$ if there exists a map $\lambda : G \to A$ such that $\lambda(xy) = \omega(x,y)\omega'^{-1}(x,y)\lambda(x)\lambda(y)$.

This second cohomology group classifies central extensions of G by A, because equivalent 2-cocycles lead to equivalent extensions.

To completely understand the role of central extensions in conformal field theory or in general, in QFT, we would need some knowledge of Hilbert spaces, which we will treat later.

But using only the notion of central extensions, we can understand the Virasoro algebra, an object of central importance in CFT.