Modular Forms

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Notes from a seminar talk. The main reference is [1].

1 The modular group

Let $\mathbb{H}$ denote the upper half of the complex plane $\mathbb{C}$, or in other words, the complex numbers with imaginary part > 0. The group

$$\mathrm{SL}_2(\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | ad - bc = 1\}. \tag{1.1}$$

acts on the upper half plane by the Mobius transformation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}, \tag{1.2}$$

It is easy to check that $\mathbb{H}$ is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$. Note that

$$- Iz = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z = \frac{-z}{-1} = z, \tag{1.3}$$

or more generally, changing the sign of an element of $\mathrm{SL}_2(\mathbb{R})$ does not change what it does to an element of the upper-half plane, so we define

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) / \pm I, \tag{1.4}$$

whose action on $\mathbb{H}$ is faithful, meaning if for $g \in \mathrm{PSL}_2(\mathbb{R})$, $gz = z$ the $g = I$. Let $\mathrm{SL}_2(\mathbb{Z})$ be the discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ that has entries in $\mathbb{Z}$. The modular group is defined as the group

$$\Gamma = \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) / \pm I \tag{1.5}$$

Let $t = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ such that $s^2 = 1, st^3 = 1$. It can be shown that s and t with these relations generate Γ.

The subset D of $\mathbb{H}$, consisting of points z such that $|z| \geq 1$ and $|\mathrm{Re}(z)| \leq 1/2$, is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$.

2 Modular functions

Definition 2.1. Let $k \in \mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}$ is called *weakly modular* of weight $2k$ if it is meromorphic on $\mathbb{H}$ such that

$$f(z) = (cz+d)^{-2k} f(\frac{az+b}{cz+d}), \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.1}$$

Let g be the image of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ. Note that

$$\frac{d(gz)}{dz} = (cz+d)^{-2} \tag{2.2}$$

which lets us write (2.1) in the form

$$\frac{f(gz)}{f(z)} = (\frac{d(gz)}{dz})^{-k} \tag{2.3}$$

or equivalently

$$f(gz)d(gz)^k = f(z)dz^k. \tag{2.4}$$

This shows that the "differential form of weight k" is invariant under Γ. Since Γ is generated by s and t, it is enough to check that it is invariant under s and t. Under the action of t,

$$f(z) = f(z+1) \tag{2.5}$$

and under the action of s

$$f(z) = z^{-2k} f(\frac{-1}{z}). \tag{2.6}$$

If condition (2.5) is satisfied, then f is a periodic function of period 1, and may written as a function $\tilde{f}(q)$ where the variable $q = \exp(2\pi i z)$. If $\tilde{f}$ can be holomorphically (meromorphically) continued at the origin, then we will say that f is holomorphic (meromorphic) at infinity. This means $\tilde{f}$ admits a Laurent expansion in the neighbourhood of the origin.

$$\tilde{f}(q) = \sum_{n=-\infty}^{\infty} a_n q^n \tag{2.7}$$

where $a_n = 0$ for n small enough.

Definition 2.2. 1. A weakly modular function is *modular* if it is meromorphic at ∞.

2. A *modular form* is a modular function that is holomorphic everywhere (including ∞). If it is zero at ∞ then it is called a cusp form (*forme parabolique* in French, *Spitzenform* in German).

A modular form of weight $2k$ is given by a series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n \exp(2\pi i n z) \tag{2.8}$$

which converges for $|q| \leq 1$ and satisfies

$$f(-1/z) = z^{2k} f(z) \tag{2.9}$$

and it is parabolic if $a_0 = 0$. The exponent is always chosen to be of the form $2k$, because if we choose it as $l \notin 2\mathbb{Z}$, then for $g = -I$ in the definition

$$f(z) = (-1)^l f(z) \tag{2.10}$$

which means $f(z)$ is the constant function with value 0.

If f and f' are modular forms of weight $2k$ and $2k'$, then ff' is a modular form of weight $2k + 2k'$.

3 Lattice functions

A lattice is a discrete subgroup G of a finite dimensional $\mathbb{R}$-vector space V such that V/G is compact, or equivalently, there exists an $\mathbb{R}$-basis of V which is a $\mathbb{Z}$ basis of G. For example, $\mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$, because $\mathbb{R}^2/\mathbb{Z}^2$ is compact (or alternatively, the basis $\{(1,0),(0,1)\}$ is a $\mathbb{Z}$ basis for $\mathbb{Z}^2$). Let $\mathcal{L}$ be the set of lattices in $\mathbb{C}$ considered as $\mathbb{R}^2$, and M the set of couples (ω_1, ω_2) of elements in $\mathbb{C}^\times$ such that Im $(\omega_1/\omega_2) > 0$. Each element of M gives rise to a lattice

$$L(\omega_1, \omega_2) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \tag{3.1}$$

with basis $\{\omega_1, \omega_2\}$. This define a surjection $M \to \mathcal{L}$.

If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ acts on the vector $(\omega_1, \omega_2) \in M$, we have $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$, which is also (obviously) a basis for $L(\omega_1, \omega_2)$.

Proposition 3.1. *Two elements of M define the same lattice if and only if they are congruent mod* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Suppose (ω_1, ω_2) and (ω_1', ω_2') define the same lattice, then there exists a integer matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ having determinant ± 1 that maps the first basis to the second. If $\det(g) = -1$, then $\mathrm{Im}(\omega_1/\omega_2)$ and $\mathrm{Im}(\omega_1'/\omega_2')$, which is not possible by definition, so $\det(g) = 1$. $\square$

There is thus an bijection between the set $\mathcal{L}$ of lattices in $\mathbb{C}$ and the quotient $M/\mathrm{SL}_2(\mathbb{Z})$. If we quotient M by scalings in $\mathbb{C}^\times$, then this is in bijection with $\mathbb{H}$ (via $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2$) and the action of $\mathrm{SL}_2(\mathbb{Z})$ reduces to that of Γ.

Proposition 3.2. *The map $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2$ induces a bijection between $\mathcal{L}/\mathbb{C}^\times$ and H/Γ.*

Remark 3.3. The set $\mathcal{L}/\mathbb{C}^\times \simeq H/\Gamma$ also admits an alternative description as the set of isomorphism classes of elliptic curves.

A function F *has weight* $2k$ for $k \in \mathbb{Z}$ if

$$F(\lambda L) = \lambda^{-2k} F(L) \tag{3.2}$$

for a lattice L and $\lambda \in \mathbb{C}^*$. Denoting the value of F on $L(\omega_1, \omega_2)$ by $F(\omega_1, \omega_2)$, the above condition yields

$$F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2k} F(\omega_1, \omega_2) \tag{3.3}$$

and further $F(\omega_1, \omega_2)$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on M. The product $\omega_2^{2k} F(\omega_1, \omega_2)$ only depends on the ratio $z = \omega_1/\omega_2$, and so there exists a function

$$F(\omega_1, \omega_2) = \omega_2^{-2k} f(\omega_1/\omega_2) \tag{3.4}$$

Since F is invariant under $\mathrm{SL}_2(\mathbb{Z})$, we get

$$f(z) = (cz+d)^{-2k} f(\frac{az+b}{cz+d}) \tag{3.5}$$

Conversely, for every f satisfying this condition, we can construct a lattice function using (3.4). There is thus an identification of modular functions of weight $2k$ with certain lattice functions of weight $2k$.

4 Eisenstein series

Lemma 4.1. *For a lattice L in $\mathbb{C}$, the series $\sum_{\gamma \in L}{}' 1/|\gamma|^\sigma$ converges for $\sigma > 2$, where the $'$ denotes the sum over non-zero elements of L.*

Let $k > 1$. If L is a lattice in $\mathbb{C}$, define

$$G_k(L) = \sum_{\gamma \in L}{}' 1/\gamma^{2k} \tag{4.1}$$

This series converges because of the lemma above, and G_k is clearly of weight $2k$. G_k is called the Eisenstein series of index k. It can be seen as a function defined on M given by

$$\mathrm{G}_k(\omega_1, \omega_2) = \sum_{m,n}{}' \frac{1}{(m\omega_1 + n\omega_2)^{2k}}. \tag{4.2}$$

Fact 4.2. *If $k > 1$, the Eisenstein series $\mathrm{G}_k(z)$ is a modular form of weight 2k, and*

$$\mathrm{G}_k(\infty) = 2\zeta(2k) \tag{4.3}$$

where ζ is the Riemann zeta function.

The lowest weight Eisenstein series G_2 and G_3 have weights 4 and 6 respectively. It is standard to define

$$g_2 = 60\mathrm{G}_2, \ g_3 = 140\mathrm{G}_3 \tag{4.4}$$

and we get $g_2(\infty) = 120\zeta(4) = \frac{4}{3}\pi^4$ and $g_3(\infty) = 280\zeta(6) = \frac{8}{27}\pi^6$. The Eisenstein series thus give us a large class of examples of modular forms. If we define

$$\Delta = g_2^3 - 27g_3^2, \tag{4.5}$$

we see that $\Delta(\infty) = 0$ and this is an example of a cusp form.

References

[1] J. Serre. *Cours d'arithmétique: par Jean-Pierre Serre*. SUP. Le mathématicien. Presses universitaires de France, 1970.